CHAPTER II

UNIVERSALITY AND A FUNDAMENTAL LEMMA

1. Introduction

The subject of this and the following chapters has its origin in a theorem of H. Bohr and R. Courant [1] which belongs to the classical value distribution theory of the Riemann zeta-function, $\zeta(s)$. (An account of this theory may be found in Titchmarch [19; Chap. 11].) We have in mind the

Theorem A. For fixed $\sigma \in (\frac{1}{2}, 1)$, the curve $\gamma(\tau) = \zeta(\sigma + i\tau)$ is dense in $\mathbb{C}$.

Recently S.M. Voronin [20] strengthened Theorem A by proving the following two theorems:

Theorem B. For fixed $\sigma \in (\frac{1}{2}, 1)$, the curve $\gamma_1(\tau) = (\zeta(\sigma + i\tau), \zeta'(\sigma + i\tau), \ldots, \zeta^{(n-1)}(\sigma + i\tau))$ is dense in $\mathbb{C}^n$.

Theorem C. For fixed $s_1, s_2, \ldots, s_n$ with $\operatorname{Re} s_j \in (\frac{1}{2}, 1)$ for $1 \leq j \leq n$ and $\operatorname{Im} s_j \neq \operatorname{Im} s_k$ for $j \neq k$, the curve $\gamma_2(\tau) = (\zeta(s_1 + i\tau), \zeta(s_2 + i\tau), \ldots, \zeta(s_n + i\tau))$ is dense in $\mathbb{C}^n$.

Note that Theorem A is contained in both Theorem B and Theorem C.

Interesting as Theorems B and C are, Voronin [21] subsequently extended Theorem B to show that $\zeta(s)$ is "universal":

Theorem D. Let D_r be the closed disc of radius $r < \frac{1}{4}$ centered at $s = \frac{3}{4}$ in the complex plane. Suppose $f(s)$ is analytic on the interior of D_r and continuous and nonvanishing on D_r. Then for any $\varepsilon > 0$, there is a $\tau \in \mathbb{R}$ such that

$$\max_{s\varepsilon D_r} |\zeta(s + i\tau) - f(s)| < \varepsilon .$$

This is a universality theorem in that it asserts that the translates of $\zeta(s)$ approximate <u>all</u> the functions in a large class of functions. We may also consider Theorem D to be an infinite dimensional Hilbert space analogue of Theorem B. In fact, it is not difficult to see that Theorem D implies Theorem B.

The condition in Theorem D that $f(s)$ be nonvanishing on D_r is significant. Indeed, suppose that the conclusion of Theorem D were true for a function $f(s)$ which has a zero interior to D_r but not on the boundary of D_r, ∂D_r. Let $\min_{s\varepsilon\partial D_r} |f(s)| = m > 0$ and choose $\varepsilon = m$ in Theorem D. Then there would exist a $\tau \in \mathbb{R}$ such that

$$\max_{s\varepsilon D_r} |\zeta(s + i\tau) - f(s)| < \min_{s\varepsilon \partial D_r} |f(s)| .$$

This and Rouché's theorem would imply that $\zeta(s)$ has a zero in the disc $D_r + i\tau$. In other words, it would follow that the Riemann hypothesis is false.

In the following chapters our object will be to prove other universality theorems. We now describe some of our main results.

Voronin [21] mentions that Theorem D is true for all Dirichlet L-functions, $L(s,\chi)$. In Chapter III we show that much more is true. Let $\varepsilon > 0$, let $q \geq 1$ be an integer, and let C be a simply connected compact set in the strip $\frac{1}{2} < \sigma < 1$. Suppose that for each character $\chi \pmod q$, $f_\chi(s)$ is continuous on C and analytic on the interior of C. Then there exists a $\tau \varepsilon \mathbb{R}$ such that

$$\max_{s\varepsilon C} |L(s + i\tau, \chi) - e^{f_\chi(s)}| < \varepsilon$$

for each $\chi \pmod q$. As applications, we deduce a universality analogue of Theorem C and we prove that the Dedekind zeta-function of any abelian extension of the rationals is universal.

In Chapter IV we prove that certain Hurwitz zeta-functions, $\zeta(s,\alpha)$, are universal. In particular, suppose that $\alpha \varepsilon (0, 1)$, $\alpha \neq \frac{1}{2}$, and α is rational. Again we denote by C any simply connected compact set in the strip

$\frac{1}{2} < \sigma < 1$. If $\varepsilon > 0$ and $f(s)$ is continous on C and analytic on the interior of C , there is a $\tau \in \mathbb{R}$ for which

$$\max_{s\varepsilon C} |\zeta(s + i\tau, \alpha) - f(s)| < \varepsilon .$$

If we take C to be a disc and choose an $f(s)$ having zeros inside C but not on the boundary of C , then we conclude from Rouché's theorem as before that $\zeta(s,\alpha)$ has zeros inside $C + i\tau$. Thus $\zeta(s, \alpha)$ has zeros off the line $\sigma = \frac{1}{2}$ but in the critical strip. The same results hold when $\alpha \varepsilon (0, 1)$ and α is transcendental, but we leave untreated the difficult case in which α is an algebraic irrational number.

In Chapter V we prove a q-analogue of Theorem D. Let C , $f(s)$, and ε be as in the previous paragraph. We show that there is a number $q_0 = q_0(C, f, \varepsilon)$ such that for every $q \geq q_0$, there exists a character χ (mod q) for which

$$\max_{s\varepsilon C} |L(s,\chi) - e^{f(s)}| < \varepsilon .$$

A. Good [6] has combined Voronin's work with a method of H.L. Montgomery [13] for proving Ω-theorems for log $|\zeta(s)|$. The results of this fusion are quite interesting. For example, Good is able to tell how large τ must be taken in order to ensure that the set $\zeta(D_r + i\tau)$ covers a prescribed annulus. Since Good desired quantitative results, it was necessary for him to develop methods which are somewhat different from Voronin's. In our proofs, we follow Good's approach rather than Voronin's.

Essentially, there are two steps to this approach. These are perhaps best described by sketching the proof that $\zeta(s)$ is universal. Let C, $f(s)$, and ε be as above. The first step is to show that if μ is sufficiently large, there is a $\rho > \mu$ and there are real numbers θ_p (p prime, $\mu < p < \rho$) such that

$$\max_{s \in C} \left| \sum_{\mu<p<\rho} e(\theta_p) p^{-s} - f(s) \right| < \frac{\varepsilon}{2} .$$

This follows from Lemma 2.2, the so-called fundamental lemma. The second step of Good's approach consists of using mean-value theorems, a zero-density estimate for $\zeta(s)$, and a general form of Kronecker's theorem on diophantine approximation, to show that there is a $\tau \in \mathbb{R}$ such that

$$\max_{s \in C} \left| \log \zeta(s + i\tau) - \sum_{\mu<p<\rho} e(\theta_p) p^{-s} \right| < \frac{\varepsilon}{2} .$$

(Of course, the logarithm must be suitably defined.) By combining both these inequalities we find that there is a $\tau \in \mathbb{R}$ for which

$$\max_{s \in C} \left| \log \zeta(s + i\tau) - f(s) \right| < \varepsilon .$$

We conclude from this that there is a $\tau \in \mathbb{R}$ which makes

$$\max_{s \varepsilon C} |\zeta(s + i\tau) - e^{f(s)}|$$

smaller than any preassigned positive value, as desired.

While the author was writing this dissertation, Voronin [22] announced that the Epstein zeta-functions and the Hurwitz zeta-functions, $\zeta(s,\alpha)$, with α rational, $\alpha \varepsilon (0, 1)$, and $\alpha \neq \frac{1}{2}$, have zeros in the right half of the critical strip. The author then obtained a copy of Voronin's dissertation abstract [25]. In it, Voronin states a simultaneous universality theorem for n-tuples of Dirichlet L-functions analogous to our Theorem 3.1. Presumably, Voronin is also able to prove a universality theorem for $\zeta(s,\alpha)$ with α rational. However, it seems that he has not duplicated either our universality theorem for $\zeta(s,\alpha)$ with α transcendental, or the q-analogue of Chapter V. Furthermore, all of Voronin's univesality theorems involve the discs D_r whereas ours involve arbitrary compact sets. As we shall see, this fact has some interesting consequences (see Theorem 3.3, for example). Finally, where it has been possible to compare proofs, they differ significantly.

§2. Some Notation and Statement of the Fundamental Lemma

In Section II.1 we mentioned that the first part of our method for proving universality theorems is based on the fundamental lemma. Before we can state this lemma we must introduce our notation.

Throughout this chapter, U denotes an open bounded rectangle with vertices on the lines $\sigma = \sigma_1$ and $\sigma = \sigma_2$ where $\frac{1}{2} < \sigma_1 < \sigma_2 < 1$. By $L_2(U)$ we mean the set of equivalence classes of all complex-valued functions which are square integrable on U with respect to Lebesque measure. This is a Hilbert space with the inner product given by

$$\langle f_1(s), f_2(s) \rangle = \int_U f_1(s)\overline{f_2}(s)\, d\sigma dt$$

for $f_1(s)$ and $f_2(s) \varepsilon L_2(U)$. We denote by $P_2(U)$ the closed subspace of $L_2(U)$ generated by the polynomials in s.

Let $\Lambda = \{\lambda\}$ be a monotone increasing sequence of real numbers tending to ∞. We write $N_\Lambda(x)$ for its counting function. Thus

$$N_\Lambda(x) = \sum_{\substack{\lambda \le x \\ \lambda \varepsilon \Lambda}} 1 .$$

If $0 < \mu < \rho$, we let $\Gamma_{\mu,\rho}$ be the set of all functions of the form

$$\sum_{\substack{\mu < e^\lambda \le \rho \\ \lambda \varepsilon \Lambda}} z_\lambda e^{-\lambda s} ,$$

where the z_λ are complex numbers with $|z_\lambda| \le 1$. Each

$\Gamma_{\mu,\rho}$ is a convex set. Also, if $\mu < \rho_1 < \rho_2$, then

$$\Gamma_{\mu,\rho_1} \subseteq \Gamma_{\mu,\rho_2} .$$

Hence

$$\Gamma_\mu = \bigcup_{\mu<\rho} \Gamma_{\mu,\rho}$$

is a convex set. Finally, since the functions in Γ_μ are entire and U is a bounded set, we conclude that Γ_μ is a convex subset of $L_2(U)$.

The proof of the fundamental lemma is based on

Lemma 2.1. Suppose that for any fixed $c > 0$, we have

$$|N_\Lambda(x \pm \frac{c}{x^2}) - N_\Lambda(x)| \gg \frac{e^x}{x^3} .$$

Then for any $\mu > 0$, the closure of Γ_μ in $L_2(U)$ is $P_2(U)$.

Remark. Our hypothesis on $N_\Lambda(x)$ in Lemma 2.1 is far from being the most general one possible. We have just made it general enough to cover the cases we shall require.

When Λ is the sequence of logarithms of the primes, Lemma 2.1 yields Theorem 4 (ii) of A. Good [6]. Voronin [21] deduces his fundamental lemma from a theorem of D.V. Pečerskiĭ [15]. Basically, Pečerskiĭ's theorem is an extension to Hilbert space of Riemann's theorem on rearrangements of conditionally convergent series of real numbers.

We now state

Lemma 2.2 (The Fundamental Lemma). Suppose that

$$N_\Lambda(x) \ll e^x$$

and that for any fixed $c > 0$,

$$|N_\Lambda(x \pm \frac{c}{x^2}) - N_\Lambda(x)| \gg \frac{e^x}{x^3} .$$

Let C be a simply connected compact set in the strip $\frac{1}{2} < \sigma_1 < \sigma < \sigma_2 < 1$ and suppose $f(s)$ is continuous on C and analytic in the interior of C . Then for each $\mu > 0$ there exists a constant $\rho_0 = \rho_0(\sigma_1, \sigma_2, C, \Lambda, f, \mu)$ such that if $\rho \geq \rho_0$, there are numbers $\theta_\lambda \in \mathbb{R}$ for which

$$\max_{s \in C} |f(s) - \sum_{\substack{\mu < e^\lambda \leq \rho \\ \lambda \in \Lambda}} e(\theta_\lambda) e^{-\lambda s}| \ll \mu^{-\frac{1}{2}} .$$

The constant implied by $\ll$ depends only on σ_1 , σ_2 , C , and Λ .

In the next section we set down the lemmas needed for the proofs of Lemmas 2.1 and 2.2.

§3. Auxiliary Lemmas

Lemma 2.3. Let $P(x)$ be a polynomial of degree n. Let $a \in \mathbb{R}$ and suppose that

$$\max_{a-1 \le x \le a+1} |P(x)| \le M .$$

Then

$$\max_{a-1 \le x \le a+1} |P'(x)| \le n^2 M .$$

This is a classical result of Markov [11] (see Cheney [3; p. 91]).

The next lemma is Mergelyan's theorem (see Rudin [17; Theorem 20.5]).

Lemma 2.4. If C is a compact set in the plane whose complement is connected, if $f(s)$ is a continuous complex function on C which is analytic in the interior of C, and if $\varepsilon > 0$, then there exists a polynomial $P(s)$ such that

$$\max_{s \in C} |f(s) - P(s)| < \varepsilon .$$

Lemma 2.5. Let C be a compact subset of the rectangle U. Let

$$d = \min_{z \in \partial U} \min_{s \in C} |s - z| .$$

If $f(s)$ is analytic on U and

$$\int_U |f(s)|^2 \, d\sigma dt \le \varepsilon ,$$

then

$$\max_{s \in C} |f(s)| \le \frac{\sqrt{\varepsilon/\pi}}{d} .$$

<u>Proof</u>: Let s_0 be any point in C and let D_R denote the closed disc centered at s_0 with radius $R < d$. Then $D_R \subset U$. Now for each r with $0 \le r \le R$, we have

$$f(s_0) = \frac{1}{2\pi} \int_0^{2\pi} f(s_0 + re^{i\theta}) d\theta .$$

Hence

$$\begin{aligned} \pi R^2 |f(s_0)| &= 2\pi \int_0^R |f(s_0)| r dr \\ &= 2\pi \int_0^R \left| \frac{1}{2\pi} \int_0^{2\pi} f(s_0 + re^{i\theta}) d\theta \right| r dr \\ &\le \int_0^{2\pi} \int_0^R |f(s_0 + re^{i\theta})| \; r dr d\theta \\ &\le \left(\int_0^{2\pi} \int_0^R |f(s_0 + re^{i\theta})|^2 r dr d\theta \right)^{1/2} \left(\int_0^{2\pi} \int_0^R r dr d\theta \right)^{1/2} \end{aligned}$$

$$\leq \sqrt{\pi}\, R \left(\int_U |f(s)|^2 \, d\sigma dt \right)^{1/2}$$

$$\leq R\sqrt{\pi\varepsilon} \quad .$$

Therefore

$$|f(s_0)| \leq \frac{\sqrt{\varepsilon/\pi}}{R} \quad .$$

The result follows from this. □

The following lemma is due to A. Good [6; Lemma 6].

Lemma 2.6. Let K and L be positive integers with $K \leq L$. Let $a_{k\ell}$ and b_k , $1 \leq k \leq K$, $1 < \ell \leq L$ be complex numbers. Suppose the system of equations

$$(1) \qquad \sum_{\ell=1}^{L} a_{k\ell} w_\ell = b_k \, , \qquad 1 \leq k \leq K$$

has a solution $\vec{w} \in D_L$, where $D_L = \{(w_1, \ldots, w_L) | w_\ell \in \mathbb{C}, |w_\ell| \leq 1 \, , \, 1 \leq \ell \leq L\}$. Then the system also has a solution $\vec{w'} \in D_L$ for which $|w'_\ell| = 1$ for at least $L - K$ integers $\ell \leq L$.

<u>Proof</u>: The proof is by induction on L . The assertion is trivial for $K = L$. For $1 \leq K \leq L$, the solutions of (1) form a linear manifold of dimension at least $L - K$ over the complex numbers. By hypothesis, this manifold intersects

D_L . Hence it intersects the boundary of D_L . That is, there exists a real number θ such that the sytem

$$\sum_{\ell=1}^{L-1} a_{k\ell} w_\ell + a_{kL} e^{i\theta} = b_k \,, \qquad 1 \le k \le K$$

has a solution in D_{L-1} . Since this is a system of K equations in $L - 1$ unknowns, the inductive hypothesis asserts that it has a solution $(w_1', \ldots, w_{L-1}') \in D_{L-1}$ with $|w_\ell'| = 1$ for at least $L - K - 1$ integers ℓ with $1 \le \ell \le L-1$. This proves the lemma. □

Lemma 2.7. Let C be a compact set in the strip $\frac{1}{2} < \sigma_1 < \sigma < \sigma_2 < 1$. Let

$$N_\Lambda(x) \ll e^x \,.$$

Assume that $0 < \mu < \rho$ and that for each $\lambda \in \Lambda$ with $\mu < e^\lambda \le \rho$ we have $|z_\lambda| \le 1$. Then there are numbers $\theta_\lambda \in \mathbb{R}$ such that

$$\max_{s\in C} \Big| \sum_{\substack{\mu<e^\lambda\le\rho \\ \lambda\in\Lambda}} z_\lambda e^{-\lambda s} - \sum_{\substack{\mu<e^\lambda\le\rho \\ \lambda\in\Lambda}} e(\theta_\lambda) e^{-\lambda s} \Big| \ll \mu^{-1/2} \,.$$

The implied constant depends only on σ_1 , C , and Λ .

Proof: Choose $A \ge 1$ so that $|s| \le A$ for all $s \in C$, let U be the open rectangle with vertices $\sigma_1 \pm iA$,

$\sigma_2 \pm iA$. Then $C \subseteq U$. Set $K = [Ae^2 \log(2\nu+2)] + 1$ for $\nu > 0$. Let $\nu < \nu' \leq 2\nu$. We have

$$\sum_{\substack{\nu<e^\lambda\leq\nu' \\ \lambda\in\Lambda}} z_\lambda e^{-\lambda s} = \sum_{\substack{\nu<e^\lambda\leq\nu' \\ \lambda\in\Lambda}} z_\lambda \left(\sum_{k=0}^{K} \frac{(-\lambda s)^k}{k!} \right)$$

$$+ \sum_{\substack{\nu<e^\lambda\leq\nu' \\ \lambda\in\Lambda}} z_\lambda \left(\sum_{k>K} \frac{(-\lambda s)^k}{k!} \right) .$$

Since $N_\Lambda(x) \ll e^x$, the number of $\lambda \varepsilon \Lambda$ with $e^\lambda \leq \nu'$ is

$$\ll \nu' \ll \nu .$$

Thus, if $|z_\lambda| \leq 1$, we find that

$$\sum_{\substack{\nu<e^\lambda\leq\nu' \\ \lambda\in\Lambda}} z_\lambda \left(\sum_{k>K} \frac{(-\lambda s)^k}{k!} \right) \ll \nu \sum_{k>K} \frac{(A \log(2\nu+2))^k}{k!}$$

$$\ll \nu \sum_{k>K} \left(\frac{Ae \log(2\nu+2)}{k}\right)^k$$

$$\ll \nu \sum_{k>K} \left(\frac{Ae \log(2\nu+2)}{K}\right)^k$$

for $s \varepsilon U$. (Here we have used the inequality $\frac{1}{k!} \leq \left(\frac{e}{k}\right)^k$ when $k \geq 1$.) The last sum is the tail of a geometric series with ratio <1 so the entire expression is

$$\ll \nu \left(\frac{Ae \log(2\nu+2)}{K}\right)^K$$

$$\ll \nu e^{-K} \ll \nu^{-1} .$$

We now see that if the z_λ each have modulus ≤ 1, then

$$(2) \quad \sum_{\substack{\nu < e^\lambda \leq \nu' \\ \lambda \varepsilon \Lambda}} z_\lambda e^{-\lambda s} = \sum_{k=0}^{K} s^k \sum_{\substack{\nu < e^\lambda \leq \nu' \\ \lambda \varepsilon \Lambda}} z_\lambda \frac{(-\lambda)^k}{k!} + O(\nu^{-1})$$

uniformly for $s \varepsilon U$. Assume that the number of $\lambda \varepsilon \Lambda$ with $\nu < e^\lambda \leq \nu'$ is at least $K + 1$. Applying Lemma 2.6 to the system

$$\sum_{\substack{\nu < e^\lambda \leq \nu' \\ \lambda \varepsilon \Lambda}} z_\lambda \frac{(-\lambda)^k}{k!} = b_k , \qquad 0 \leq k \leq K ,$$

we conclude that there are complex numbers z'_λ with $|z'_\lambda| \leq 1$ and $|z'_\lambda| < 1$ for at most $K + 1$ of the λ such that

$$\sum_{\substack{\nu < e^\lambda \leq \nu' \\ \lambda \varepsilon \Lambda}} z_\lambda \frac{(-\lambda)^k}{k!} = \sum_{\substack{\nu < e^\lambda \leq \nu' \\ \lambda \varepsilon \Lambda}} z'_\lambda \frac{(-\lambda)^k}{k!} , \qquad 0 \leq k \leq K .$$

Hence by (2),

$$\sum_{\substack{\nu < e^\lambda \leq \nu' \\ \lambda \varepsilon \Lambda}} z_\lambda e^{-\lambda s} = \sum_{\substack{\nu < e^\lambda \leq \nu' \\ \lambda \varepsilon \Lambda}} z'_\lambda e^{-\lambda s} + O(\nu^{-1})$$

for all $s \varepsilon U$. Therefore real numbers θ_λ can be found such that

$$\sum_{\substack{\nu < e^\lambda \leq \nu' \\ \lambda \varepsilon \Lambda}} z_\lambda e^{-\lambda s} = \sum_{\substack{\nu < e^\lambda \leq \nu' \\ \lambda \varepsilon \Lambda}} e(\theta_\lambda) e^{-\lambda s} + O(\nu^{-1}) + O(K\nu^{-\sigma_1})$$

for all $s \in U$. (Recall that the left edge of the rectangle U lies on the line $\sigma = \sigma_1$.) Since

$$K\nu^{-\sigma_1} \ll \nu^{-\sigma_1} \log(2\nu+2) \ll \nu^{-1/2} ,$$

we obtain

$$(3) \qquad \sum_{\substack{\nu<e^{\lambda}\leq\nu' \\ \lambda\in\Lambda}} z_{\lambda}e^{-\lambda s} = \sum_{\substack{\nu<e^{\lambda}\leq\nu' \\ \lambda\in\Lambda}} e(\theta_{\lambda})e^{-\lambda s} + O(\nu^{-1/2}) .$$

Now assume that the number of $\lambda \in \Lambda$ with $\nu < e^{\lambda} \leq \nu'$ is at most K. Then for all $s \in U$ we find that

$$\sum_{\substack{\nu<e^{\lambda}\leq\nu \\ \lambda\in\Lambda}} z_{\lambda}e^{-\lambda s} \ll K\nu^{-\sigma_1} \ll \nu^{-1/2} ,$$

and

$$\sum_{\substack{\nu<e^{\lambda}\leq\nu' \\ \lambda\in\Lambda}} e(\theta_{\lambda})e^{-\lambda s} \ll K\nu^{-\sigma_1} \ll \nu^{-1/2} ,$$

where the θ_{λ} are arbitrary real numbers. Thus (3) holds in this case also. Adding the respective sides of (3) together with $\nu' = 2\nu$ and ν of the form $\nu = 2^n$ for $0 \leq n \leq \log(\frac{\rho}{\mu})/\log 2$, and then using (3) once more if necessary to cover any remaining part of the range $(\mu,\rho]$, we obtain

$$\sup_{s \in U} \left| \sum_{\substack{\mu < e^{\lambda} \leq \rho \\ \lambda \in \Lambda}} z_{\lambda} e^{-\lambda s} - \sum_{\substack{\mu < e^{\lambda} \leq \rho \\ \lambda \in \Lambda}} e(\theta_{\lambda}) e^{-\lambda s} \right| \ll \mu^{-1/2} .$$

It is clear from the proof that the constant implied by $\ll$ depends only on Λ , σ_1 , and A . Since A depends on C , the constant depends on Λ, σ_1 , and C . The result now follows.

§4. Proof of Lemma 2.1

Let $\overline{\Gamma}\mu$ be the closure in $L_2(U)$ of $\Gamma\mu$. Since the functions in $\Gamma\mu$ are entire, $\Gamma\mu \subseteq P_2(U)$. Since $P_2(U)$ is closed in $L_2(U)$, $\overline{\Gamma}\mu \subseteq P_2(U)$. We must show that $P_2(U) \subseteq \overline{\Gamma}\mu$. As a closed subspace of the Hilbert space $L_2(U)$, $P_2(U)$ is isomorphic to its dual. Furthermore, $\overline{\Gamma}\mu$ is convex since $\Gamma\mu$ is. Therefore, by a consequence of the theorem of Hahn-Banach (see Rudin [18; Theorem 3.4]), an element $g(s) \in P_2(U)$ is in $\overline{\Gamma}\mu$ if and only if

$$\langle g(s), f(s) \rangle \in \langle \overline{\Gamma}\mu, f(s) \rangle$$

for each $f(s) \in P_2(U)$, where

$$\langle \overline{\Gamma}\mu, f(s) \rangle = \{\langle h(s), f(s) \rangle \mid h \in \overline{\Gamma}\mu\} .$$

If $f(s) = 0$, this condition is clearly satisfied. Thus, from now on we assume $f(s) \not\equiv 0$. It then suffices to show that

$$\langle \overline{\Gamma}\mu,\ f(s) \rangle = \mathbb{C}\ .$$

Now consider the projections

$$(4)\ \langle \Gamma_{\mu,\rho},\ f(s) \rangle = \left\{ \sum_{\substack{\mu < e^{\lambda} \leq \rho \\ \lambda \in \Lambda}} z_{\lambda} \int_U \overline{f}(s) e^{-\lambda s}\, d\sigma dt \,\middle|\, |z_{\lambda}| \leq 1 \right\}$$

for $\rho > \mu$. These are closed discs centered at the origin with radii

$$r_f(\mu,\rho) = \sum_{\substack{\mu < e^{\lambda} \leq \rho \\ \lambda \in \Lambda}} \left| \int_U \overline{f}(s) e^{-\lambda s}\, d\sigma dt \right| .$$

We will show that there is a sequence of ρ_j tending to ∞ with j such that the discs $\langle \Gamma_{\mu,\rho_j};\ f(s) \rangle$, which are nested, have radii tending to ∞ . It will then follow that $\langle \overline{\Gamma}\mu,\ f(s) \rangle = \mathbb{C}$ and the lemma will be proved.

To this end, define

$$(5) \qquad F(z) = \int_U \overline{f}(s) e^{(\sigma_2+\delta-s)z}\, d\sigma dt\ .$$

Clearly $F(z)$ is entire. By the Cauchy-Schwartz inequality,

$$|F(z)| \leq e^{(\sigma_2+\delta+A)|z|} \left(\int_U |f(s)|^2\, d\sigma dt \int_U d\sigma dt \right)^{1/2}$$

where $A = \sup_{s \in U} |s| < \infty$. Thus $F(z)$ is of exponential type.

Expanding the exponential in the integrand in (5) into a power series gives

$$F(z) = e^{(\sigma_2+\delta)z} \int_U \overline{f}(s) \sum_{k=0}^{\infty} \frac{(-sz)^k}{k!} \, d\sigma dt$$

$$= e^{(\sigma_2+\delta)z} \sum_{k=0}^{\infty} \frac{(-z)^k}{k!} \int_U \overline{f}(s) \, s^k \, d\sigma dt \, .$$

(The inversion of integration and summation is justified by absolute convergence.) We write this as

(6) $$F(z) = e^{(\sigma_2+\delta)z} \sum_{k=0}^{\infty} F_k \frac{(-z)^k}{k!} \quad ,$$

where

(7) $$F_k = \int_U \overline{f}(s) \, s^k \, d\sigma dt \qquad (k = 0, 1, \ldots) \quad .$$

Again by (5) we have

$$|F(x)| \le e^{\delta x} \int_U |f(s)| e^{(\sigma_2-\sigma)x} \, d\sigma dt$$

$$\le e^{\delta x} \left(\int_U |f(s)|^2 \, d\sigma dt \int_U e^{2(\sigma_2-\sigma)x} \, d\sigma dt \right)^{1/2}$$

for $x \in \mathbb{R}$. Since $\sigma < \sigma_2$ when $s \in U$, this gives

(8) $$|F(x)| \ll e^{\delta x} \qquad \text{for} \quad x \le 0 \; .$$

Now assume there is a $\delta > 0$ and an $x_0 > 0$ such that

(9) $$|F(x)| \ll e^{-\delta x} \qquad \text{for} \quad x \geq x_0 .$$

From (8) and (9) it follows that

$$\hat{F}(w) = \int_{-\infty}^{\infty} F(x)\, e^{ixw}\, dx$$

is analytic in the strip $|v| < \delta$, where $w = u + iv$. On the other hand, it also follows from (8) and (9) that $F(x) \varepsilon L_2(\mathbb{R})$. By one of the Paley-Wiener theorems (see Rudin [17; Theorem 19.3]) we may conclude from this and the fact that $F(z)$ is of exponential type that $\hat{F}(w)$ has compact support. Since $\hat{F}(w)$ is analytic in $|v| < \delta$, we must have $\hat{F}(w) \equiv 0$. Thus $F(z) \equiv 0$. By (6) and (7) this means that

$$F_k = \int_U \overline{f}(s)\, s^k\, d\sigma dt = 0 \qquad (k = 0, 1, \ldots) \quad .$$

That is, either $f(s) \equiv 0$, or $f(s)$ is orthogonal to $P_2(U)$. Since $f(s) \varepsilon P_2(U)$ and we are assuming $f(s) \not\equiv 0$, (9) cannot hold. Hence, for each $\delta > 0$, there is a sequence of real numbers x_j tending to ∞ with j such that

(10) $$|F(x_j)| \geq e^{-\delta x_j} \qquad (j = 1, 2, \ldots) \quad .$$

Now assume δ is so small that

(11) $$\sigma_2 + 2\delta < 1$$

and define a sequence $\{y_j\}_{j=1}^{\infty}$ of real numbers by the condition

(12) $$\max_{|x-x_j|\leq 1} e^{-(\sigma_2+\delta)x} |F(x)| = e^{-(\sigma_2+\delta)y_j} |F(y_j)| .$$

Let

(13) $$E_K(z) = \sum_{k=0}^{K} F_k \frac{(-z)^k}{k!} \qquad (K = 0, 1, \ldots)$$

and let B satisfy

(14) $$B \log(\frac{B}{2eA}) = 1 + \delta .$$

Note that (14) implies $B > 2eA$. By (6), (7), and (13),

$$\left| e^{-(\sigma_2+\delta)x} F(x) - E_K(x) \right| \leq \sum_{k>K} |F_k| \frac{x^k}{k!}$$

$$\leq \left(\int_U |f(s)|^2 \, d\sigma dt \int_U d\sigma dt \right)^{1/2} \sum_{k>K} \frac{(Ax)^k}{k!}$$

$$\ll \sum_{k>K} \frac{(Ax)^k}{k!} .$$

Using the inequality $\frac{1}{k!} \leq (\frac{e}{k})^k$ for $k \geq 1$, we find that this last expression is

$$\leq \sum_{k>K} \left(\frac{Aex}{k}\right)^k \leq \sum_{k>K} \left(\frac{Aex}{K}\right)^k$$

$$\ll \left(\frac{Aex}{K}\right)^K ,$$

provided that $\frac{Aex}{K} \leq \frac{1}{2}$. Since $B > 2eA$, this last condition is met when $K \geq Bx$. Assuming this, we have by (14) that

$$\left(\frac{Aex}{K}\right)^K \leq \left(\frac{Ae}{B}\right)^{Bx} = e^{-xB \log B/Ae} < e^{-x(1+\delta)} .$$

Thus

$$(15) \qquad \left| e^{-(\sigma_2+\delta)x} F(x) - E_K(x) \right| \leq e^{-x}$$

for x large and $K \geq Bx$. By (12), (15), and the observation that

$$y_j-2 \leq x \leq y_j+2 \quad \text{if} \quad x_j-1 \leq x \leq x_j+1 ,$$

we obtain

$$\max_{|x-x_j|\leq 1} |E_K(x)| \leq \max_{|x-x_j|\leq 1} e^{-(\sigma_2+\delta)x} |F(x)| + e^{-y_j+2}$$

$$= e^{-(\sigma_2+\delta)y_j} |F(y_j)| + e^{-y_j+2}$$

for $K \geq B(y_j+2)$ and j large. By (10), (11), and (12),

$$e^{-(\sigma_2+\delta)y_j} |F(y_j)| \geq e^{-(\sigma_2+2\delta)x_j} \geq e^{-y_j+2}$$

for large j . Thus

$$(16) \qquad \max_{|x-x_j|\leq 1} |E_K(x)| \leq 2e^{-(\sigma_2+\delta)y_j} |F(y_j)|$$

for $K \geq B(y_j+2)$ and j large enough. Now if $|x-x_j| \leq 1$, we see by Lemma 2.3 that

$$\begin{aligned} |E_K(x) - E_K(y_j)| &= \left|\int_{y_j}^{x} E_K'(u)\,du\right| \\ &\leq |x-y_j| \max_{|x-x_j|\leq 1} |E_K'(x)| \\ &\leq |x-y_j|\, K^2 \max_{|x-x_j|\leq 1} |E_K(x)| \quad . \end{aligned}$$

In conjunction with (16) this leads to

$$(17) \quad |E_K(x) - E_K(y_j)| \leq 2|x-y_j|\, K^2\, e^{-(\sigma_2+\delta)y_j} |F(y_j)|$$

for all x with $|x-x_j| \leq 1$, provided that $K \geq B(y_j+2)$ and j is large enough. Now at least one of the intervals

$$[y_j - \tfrac{1}{4}(2\,By_j)^{-2},\; y_j] \quad \text{or} \quad [y_j,\; y_j + \tfrac{1}{4}(2\,By_j)^{-2}]$$

is contained in $[x_j-1,\ x_j+1]$ when j is large. Without loss of generality we assume the latter is true and we write

$$I_j = [y_j,\ y_j + \tfrac{1}{4}(2\,By_j)^{-2}] .$$

If we assume that $B(y_j+2) \leq K \leq 2\,By_j$ and that j is large enough, we obtain from (15) and (17) that for all $x \in I_j$,

$$\begin{aligned}
\left|e^{-(\sigma_2+\delta)x} F(x)\right| \geq \left|e^{-(\sigma_2+\delta)y_j} F(y_j)\right| & \\
&- |E_K(y_j) - e^{-(\sigma_2+\delta)y_j} F(y_j)| \\
&- |E_K(x) - E_K(y_j)| \\
&- |e^{-(\sigma_2+\delta)x} F(x) - E_K(x)| \\
\geq e^{-(\sigma_2+\delta)y_j} |F(y_j)|(1-2|x-y_j|K^2) - e^{-x} - e^{-y_j} & \\
\geq \frac{1}{2} e^{-(\sigma_2+\delta)y_j} |F(y_j)| - 2e^{-y_j} \quad . &
\end{aligned}$$

By (10), (11), and (12), the above is

$$\geq \frac{1}{2} e^{-(\sigma_2+2\delta)x_j} - 2e^{-y_j} \geq \frac{1}{2} e^{-(\sigma_2+2\delta)(y_j+1)} - 2e^{-y_j}$$

$$\geq \frac{1}{6} e^{-(\sigma_2+2\delta)y_j} \quad .$$

Thus, if $B(y_j+2) \leq K \leq 2\,By_j$, if j is large enough, and if $x \varepsilon I_j$, we have

$$(18) \qquad |e^{-(\sigma_2+\delta)x} F(x)| \geq \frac{1}{6} e^{-(\sigma_2+2\delta)y_j} .$$

Now by our hypothesis on $N_\Lambda(x)$, we see that the number of $\lambda \varepsilon \Lambda$ that are also in I_j is

$$\geq N_\Lambda(y_j + \frac{1}{4}(2\,By_j)^{-2}) - N_\Lambda(y_j)$$

$$\gg \frac{e^{y_j}}{y_j^3} .$$

Taking $\rho_j = e^{x_j+1}$, this and (18) yield for large j that

$$\sum_{\substack{\mu<e^\lambda\leq\rho_j \\ \lambda\varepsilon\Lambda}} e^{-(\sigma_2+\delta)\lambda} |F(\lambda)| \geq \sum_{\substack{\lambda\varepsilon I_j \\ \lambda\varepsilon\Lambda}} e^{-(\sigma_2+\delta)\lambda} |F(\lambda)|$$

$$\gg \frac{e^{(1-\sigma_2-2\delta)y_j}}{y_j^3} .$$

On the other hand, by (5),

$$\sum_{\substack{\mu<e^\lambda\leq\rho_j \\ \lambda\varepsilon\Lambda}} e^{-(\sigma_2+\delta)\lambda} |F(\lambda)| = \sum_{\substack{\mu<e^\lambda\leq\rho_j \\ \lambda\varepsilon\Lambda}} \left| \int_U \bar{f}(s) e^{-\lambda s} \, d\sigma dt \right|$$

$$= r_f(\mu,\rho_j) .$$

Thus we have a sequence $\rho_j \to x$ as $j \to \infty$ such that the discs

$$\left\langle \Gamma_{\mu,\rho_j}, f(s) \right\rangle$$

e radii

$$r_f(\mu,\rho_j) \gg \frac{e^{(1-\sigma_2-2\delta)y_j}}{y_j^3} .$$

.11), we find that $r_f(\mu,\rho_j)$ tends to ∞ with j .
completes the proof of the lemma. □

Proof of Lemma 2.2

Let C be a simply connected compact set in the strip $\sigma_1 < \sigma < \sigma_2 < 1$, let $f(s)$ be continuous on C and analytic in the interior of C . Set $A = 1 + \max_{s\varepsilon C} |s|$ and let U be the open rectangle with vertices $\sigma_1 \pm iA$, $\sigma_2 \pm iA$. Thus $C \subseteq U$. Given $\mu > 0$, there exists by Lemma 2.4 a polynomial $P(s)$ such that

(19) $$\max_{s\varepsilon C} |f(s) - P(s)| < \mu^{-1/2} .$$

Now since our hypotheses on $N_\Lambda(x)$ include the hypothesis of Lemma 2.1, we have $P_2(U) = \overline{\Gamma}_\mu$. Thus $P(s) \varepsilon \overline{\Gamma}_\mu$. This means there is a sequence of real numbers ρ_j tending to infinity with j and there are complex numbers $z_\lambda^{(j)}$ with

$|z_\lambda^{(j)}| \le 1$ such that

$$\int_U \Big| \sum_{\substack{\mu < e^\lambda \le \rho_j \\ \lambda \in \Lambda}} z_\lambda^{(j)} e^{-\lambda s} - P(s) \Big|^2 \, d\sigma dt \le \mu^{-1} .$$

It follows easily from this that there is a number ρ_0 depending on U, Λ, P, and μ such that for any $\rho \ge \rho_0$, there are complex z_λ with $|z_\lambda| \le 1$ for which

$$\int_U \Big| \sum_{\substack{\mu < e^\lambda \le \rho \\ \lambda \in \Lambda}} z_\lambda e^{-\lambda s} - P(s) \Big|^2 \, d\sigma dt \le \mu^{-1} . \tag{20}$$

Since $P(s)$ and $\sum_{\substack{\mu < e^\lambda \le \rho \\ \lambda \in \Lambda}} z_\lambda e^{-\lambda s}$ are both analytic on U, we obtain from (20) and Lemma 2.5 that

$$\max_{s \in C} \Big| \sum_{\substack{\mu < e^\lambda \le \rho \\ \lambda \in \Lambda}} z_\lambda e^{-\lambda s} - P(s) \Big| \le (\pi\mu)^{-1/2} d^{-1} , \tag{21}$$

where

$$d = \min_{z \in \partial U} \min_{s \in C} |s - z| .$$

Since we are assuming $N_\Lambda(x) \ll e^x$, Lemma 2.7 implies there are $\theta_\lambda \in \mathbb{R}$ such that

$$\max_{s \in C} \Big| \sum_{\substack{\mu < e^\lambda \le \rho \\ \lambda \in \Lambda}} z_\lambda e^{-\lambda s} - \sum_{\substack{\mu < e^\lambda \le \rho \\ \lambda \in \Lambda}} e(\theta_\lambda) e^{-\lambda s} \Big| \ll \mu^{-1/2} ;$$

the constant depends on σ_1 , C , Λ . Combining this with (21) yields

$$\max_{s \varepsilon C} \left| P(s) - \sum_{\substack{\mu < e^{\lambda} \leq \rho \\ \lambda \varepsilon \Lambda}} e(\theta_\lambda) e^{-\lambda s} \right| \ll \mu^{-1/2} ,$$

where the implied constant depends on σ_1 , σ_2 , C , and Λ (since d in (21) depends on σ_1 , σ_2 , and C). This and (19) imply

$$\max_{s \varepsilon C} \left| f(s) - \sum_{\substack{\mu < e^{\lambda} \leq \rho \\ \lambda \varepsilon \Lambda}} e(\theta_\lambda) e^{-\lambda s} \right| \ll \mu^{-1/2} .$$

It only remains to note that the number ρ_0 above depends on σ_1 , σ_2 , C , f , Λ , and μ , since U depends on σ_1 , σ_2 , C and the polynomial $P(s)$ depends on U , f , and μ . □

CHAPTER III

SIMULTANEOUS UNIVERSALITY OF L-FUNCTIONS

§1. Statement of Results

The principal result of this chapter is

Theorem 3.1. Let $q \geq 1$ be an integer and let C be a simply connected compact set in $\frac{1}{2} < \sigma < 1$. Suppose that for each prime $p|q$ we have $0 \leq \theta_p < 1$ and that for each character χ (mod q), $f_\chi(s)$ is continuous on C and analytic in the interior of C. If $\varepsilon > 0$, there is a $\tau \in R$ such that

$$(1) \qquad \left\| \frac{-\tau \log p}{2\pi} - \theta_p \right\| < \varepsilon \qquad (p|q)$$

and

$$\max_{s \in C} \left| L(s+i\tau, \chi) - e^{f_\chi(s)} \right| < \varepsilon \qquad (\chi \,(\mathrm{mod}\ q)) .$$

We explicitly state that a τ satisfying (1) exists because this form of the theorem lends itself most easily to the applications we have in mind.

S.M. Voronin [23] has proved that for fixed $\sigma \in (\frac{1}{2}, 1)$, the curve

$$\gamma(\tau) = (L(\sigma+i\tau, \chi_1), \ldots, L(\sigma+i\tau, \chi_n))$$

is dense in $\mathbb{C}^n$ if the characters $\chi_1, \ldots, \chi_n$ are pairwise nonequivalent. Presumably this means no two of the characters are induced by the same primitive character. Voronin [25] has also stated a universality version of this result analogous to our Theorem 3.1. Specifically, let D_r be a closed disc of radius $r < \frac{1}{4}$ centered at $s = \frac{3}{4}$ and suppose that $\varepsilon > 0$. If $\chi_1, \ldots, \chi_n$ are as above and if $f_1(s), \ldots, f_n(s)$ are continuous on D_r and analytic in the interior of D_r , then there is a $\tau \in \mathbb{R}$ such that

$$\max_{s\in D_r} |L(s+i\tau, \chi_j) - e^{f_j(s)}| < \varepsilon, \qquad 1 \le j \le n .$$

From this, Voronin deduces that certain n-tuples of Dedekind zeta-functions are simultaneously universal. Such results can also be obtained from Theorem 3.1. However, we are content to establish

Theorem 3.2. Let C be a simply connected compact set in $\frac{1}{2} < \sigma < 1$. Assume K is an abelian field extension of $\mathbb{Q}$ and let $\zeta_K(s)$ be the Dedekind zeta-function of K . If $f(s)$ is continuous on C and analytic in the interior of C , if $\varepsilon > 0$, then there is a $\tau \in \mathbb{R}$ such that

$$\max_{s \in C} |\zeta_K(s+i\tau) - e^{f(s)}| < \varepsilon .$$

To prove this, recall that there is a positive integer q such that K is a subfield of the cyclotomic field $\mathbb{Q}[e^{2\pi i/q}]$ (see Marcus [10; p. 193]). If G is the Galois group of K over $\mathbb{Q}$, then the group of characters on G, $\hat{G}$, is a subgroup of the group of Dirichlet characters (mod q). We then have that

$$(2) \qquad \zeta_K(s) = \prod_{p|q} (1 - p^{-sf_p})^{-r_p} \prod_{\chi \varepsilon \hat{G}} L(s,\chi) ,$$

where f_p is the degree of inertia of any prime $\mathfrak{p}$ in K lying over p, and $r_p = \phi(q)/f_p$ (see Marcus [10; p. 195]). Obviously the principal character (mod q), χ_0, is in $\hat{G}$. We set

$$f_\chi(s) = \begin{cases} f(s) + \sum_{p|q} r_p \log(1 - p^{-sf_p}) & \text{if } \chi = \chi_0 \\ 0 & \text{if } \chi \varepsilon \hat{G} , \chi \neq \chi_0 . \end{cases}$$

If $\varepsilon > 0$, we obtain from Theorem 3.1 that there is a $\tau \varepsilon \mathbb{R}$ such that

$$(3) \quad L(s+i\tau,\chi) = \begin{cases} e^{f(s)} \prod_{p|q} (1 - p^{-sf_p})^{r_p} + \delta_{\chi_0}(s) & \text{if } \chi = \chi_0 \\ 1 + \delta_\chi(s) & \text{if } \chi \varepsilon \hat{G} , \chi \neq \chi_0 \end{cases}$$

uniformly for $s \varepsilon C$, and

(4) $$\left\| - \frac{\tau \log p}{2\pi} \right\| < \varepsilon \qquad (p|q) ,$$

where

(5) $$\max_{s \varepsilon C} |\delta_\chi(s)| < \varepsilon \qquad (\chi \varepsilon \hat{G}) .$$

For such a τ we find from (2) and (3) that

(6) $$\zeta_K(s+i\tau) = \prod_{p|q}(1 - p^{-(s+i\tau)f_p})^{-r_p}\left(e^{f(s)} \prod_{p|q}(1 - p^{-sf_p})^{r_p} + \delta_{\chi_0}(s)\right) \cdot \prod_{\substack{\chi \neq \chi_0 \\ \chi \varepsilon \hat{G}}} (1 + \delta_\chi(s))$$

uniformly for $s \varepsilon C$. It follows from (4) and (5) that the right-hand side of (6) will be as close as we like to $e^{f(s)}$ uniformly on C, if ε is taken small enough. This proves Theorem 3.2.

The only L-function (mod 1) is the Riemann zeta-function. Thus Voronin's Theorem D of Section II.1 follows from Theorem 3.1 on taking $q = 1$. However, the fact that our set C is somewhat arbitrary allows us to state a more dramatic result.

Theorem 3.3. Let $C_1, \ldots, C_n$ be disjoint simply connected compact sets in $\frac{1}{2} < \sigma < 1$. Suppose $f_1(s), \ldots,$ $f_n(s)$ are functions such that $f_j(s)$ is continuous on C_j

and analytic in the interior of C_j, $1 \le j \le n$. If $\varepsilon > 0$, there exists a $\tau \in \mathbb{R}$ such that

$$\max_{s \in C_j} |\zeta(s+i\tau) - e^{f_j(s)}| < \varepsilon, \qquad 1 \le j \le n .$$

This is a universality analogue of Voronin's Theorem C of Section II.1. Analogous results hold for the other functions for which we have universality theorems.

The proof of Theorem 3.3 is quite simple. Let

$$C = \bigcup_{j=1}^{n} C_j .$$

Since the C_j are disjoint, there is a function $f(s)$ which is analytic in the interior of C and continuous on C such that

$$f(s) = f_j(s)$$

for $s \in C_j$, $1 \le j \le n$. Taking $q = 1$ in Theorem 3.1, we obtain a $\tau \in \mathbb{R}$ for which

$$\max_{s \in C} |\zeta(s+i\tau) - e^{f(s)}| < \varepsilon ;$$

but

$$\max_{s \in C_j} |\zeta(s+i\tau) - e^{f_j(s)}| \le \max_{s \in C} |\zeta(s+i\tau) - e^{f(s)}|$$

for $1 \le j \le n$. The result follows from this.

The remainder of this chapter is devoted to proving Theorem 3.1.

§2. Auxiliary Lemmas

Our first lemma is a weak version of a theorem of Montgomery [12; Theorem 12.1].

Lemma 3.1. Let $q \geq 1$ be an integer and let $T \geq 2$. For each character $\chi \pmod q$, let $N(\sigma,T,\chi)$ be the number of zeros $\beta+i\gamma$ of $L(s,\chi)$ with $\sigma \leq \beta \leq 1$, $0 \leq |\gamma| \leq T$. Then for $\frac{1}{2} \leq \sigma \leq 1$ we have

$$\sum_{\chi \pmod q} N(\sigma,T,\chi) \ll (qT)^{3/2-\sigma} \log^{14} qT .$$

The next lemma is classical (see Prachar [16; Chap. 7, Satz 3.3]).

Lemma 3.2. For each character $\chi \pmod q$ the number of zeros $\beta+i\gamma$ of $L(s,\chi)$ such that $|\gamma-t| \leq 1$ and $0 \leq \beta \leq 1$ is

$$\ll \log q(|t| + 2) .$$

The following is a quantitative form of the prime number theorem for arithmetic progressions (see Prachar [16; Chap. 4, Satz 7.5]).

Lemma 3.3. Let

$$\pi(x;\, q,a) = \sum_{\substack{p\le x \\ p\equiv a \pmod q \\ p \text{ prime}}} 1 \ .$$

Then for $x \geq x_0(q)$ and $(a,q) = 1$,

$$\pi(x;\, q,a) = \frac{1}{\phi(q)} \operatorname{li} x + O(x\, e^{-c\sqrt{\log x}}) \ ;$$

the constants are absolute.

Lemma 3.4. Let $\delta = \min\limits_{r\neq s}|\lambda_r-\lambda_s|$, where $\lambda_1, \ldots, \lambda_R$ are distinct real numbers. Then for any complex numbers $a_1, \ldots, a_R$,

$$\int_0^T \Bigl|\sum_{r=1}^{R} a_r e^{i\lambda_r t}\Bigr|^2 dt = (T + 2\pi\delta^{-1}\theta) \sum_{r=1}^{R} |a_r|^2 \ ,$$

where θ is real and $|\theta| \leq 1$.

This is due to Montgomery and Vaughan [14; Corollary 2].

For a proof of the next lemma see Titchmarsh [19; Sec. II.7].

Lemma 3.5. Suppose that $\lambda_1, \ldots, \lambda_K$ are linearly independent over $\mathbb{Q}$ and that $\theta_1, \ldots, \theta_K$ are fixed numbers with $0 \leq \theta_k < 1$, $1 \leq k \leq K$. For $0 < d < \frac{1}{2}$, let $I_d(T)$ be the sum of the lengths of the intervals between $\tau = 0$ and $\tau = T$ such that

$$\|-\tau\lambda_k - \theta_k\| \leq d , \qquad 1 \leq k \leq K .$$

Then

$$\lim_{T\to\infty} \frac{I_d(T)}{T} = (2d)^K .$$

Lemma 3.6. Suppose that C is a compact set in the strip $\frac{1}{2} < \sigma_1 < \sigma < \sigma_2 < 1$ and that Λ is a monotone increasing sequence of real numbers which tends to ∞, is linearly independent over $\mathbb{Q}$, and has the counting function $N_\Lambda(u) << e^u$. Let $\rho \geq e$ be so large that there is at least one $\lambda \in \Lambda$ with $e^\lambda \leq \rho$ and set

$$S(s,T) = \sum_{\substack{\rho<e^\lambda<T \\ \lambda\in\overline{\Lambda}}} b_\lambda(T)e^{-\lambda s} ,$$

where the $b_\lambda(T)$ are complex-valued functions of T with $|b_\lambda(T)| \leq 1$. Also let $\lambda_1, \ldots, \lambda_K$ be those $\lambda \in \Lambda$ with $e^\lambda \leq \rho$ and fix $0 \leq \theta_k < 1$, $1 \leq k \leq K$. If for $0 < d < \frac{1}{2}$, $\mathcal{L}_d(T)$ is the set of $\tau \in [0,T]$ satisfying

$$\left\|-\frac{\tau\lambda_k}{2\pi} - \theta_k\right\| \leq d , \qquad 1 \leq k \leq K ,$$

then for T sufficiently large

$$\int_{\ell_d(T)} \max_{s\epsilon C} |S(s+i\tau,T)|^2 dt << (2d)^K \rho^{1-2\sigma_1}(T + \delta^{-1}(T)) ,$$

where

$$\delta(T) = \min_{\substack{e^{\lambda} \neq e^{\lambda'} < T \\ \lambda,\lambda' \epsilon \overline{\Lambda}}} |\lambda-\lambda'| .$$

The constant implicit in the $<<$ symbol depends only on σ_1 , σ_2 , C , and Λ .

Proof: Let U be the interior of the rectangle with vertices $\sigma_1 \pm iA$, $\sigma_2 \pm iA$, where $A = \max_{s\epsilon C} |s|$. Set

$$\delta = \min_{z\epsilon\partial U} \min_{s\epsilon C} |s-z| .$$

By Cauchy's integral formula we have

$$\max_{s\epsilon C} |S(s+i\tau, T)|^2 = \max_{s\epsilon C} \left|\frac{1}{2\pi i} \int_{\partial U} \frac{S(z+i\tau, T)}{z - s} dz\right|^2$$

$$\leq \frac{1}{(2\pi\delta)^2} \int_{\partial U} |S(z+i\tau, T)|^2 |dz| \int_{\partial U} |dz| .$$

Thus

$$\int_{\ell_d(T)} |\max_{s\varepsilon C} S(s+i\tau,\ T)|^2\ d\tau$$

$$\leq \frac{1}{(2\pi\delta)^2} \int_{\partial U} |dz| \int_{\partial U} \left(\int_{\ell_d(T)} |S(z+i\tau,\ T)|^2\ d\tau \right) |dz|$$

$$\leq \frac{1}{(2\pi\delta)^2} \left(\int_{\partial U} |dz| \right)^2 \max_{z\varepsilon\partial U} \left(\int_{\ell_d(T)} |S(z+i\tau,\ T)|^2\ d\tau \right)$$

$$= \left(\frac{2A + (\sigma_2-\sigma_1)}{\pi\delta} \right)^2 \max_{z\varepsilon\partial U} \left(\int_{\ell_d(T)} |S(z+i\tau,\ T)|^2\ d\tau \right) .$$

Since $0 < \sigma_2-\sigma_1 < \frac{1}{2}$, we may write this as

$$(7) \int_{\ell_d(T)} \max_{s\varepsilon C} |S(s+i\tau,\ T)|^2 d\tau << \max_{z\varepsilon\partial U} \left(\int_{\ell_d(T)} |S(z+i\tau,\ T)|^2\ d\tau \right) ,$$

where the constant depends only on δ and A . An application of Cauchy's inequality to the integral on the right-hand side of (7) yields

$$(8) \int_{\ell_d(T)} |S(z+i\tau,\ T)|^2 d\tau \leq 2 \int_{\ell_d(T)} |S_1(z+i\tau,\ T)|^2\ d\tau$$

$$+ 2 \int_{\ell_d(T)} |S_2(z+i\tau,\ T)|^2 d\tau ,$$

where

$$S_1(z+i\tau,\ T) = \sum_{\substack{\rho<e^{\lambda}\leq\rho' \\ \lambda\varepsilon\Lambda}} b_\lambda(T)e^{-\lambda(z+i\tau)} ,$$

$$S_2(z+i\tau,\ T) = \sum_{\substack{\rho'<e^{\lambda}\leq T \\ \lambda\varepsilon\Lambda}} b_\lambda(T)e^{-\lambda(z+i\tau)} ,$$

and ρ' is any number greater than ρ.

First we treat the first integral on the right-hand side of (8). Fix $\varepsilon > 0$ small enough so that $0 < d-\varepsilon$ and $d+\varepsilon < \frac{1}{2}$. Define $\xi(\tau)$ to be 1 if $\|\tau\| \leq d$, 0 if $\|\tau\| \geq d+\varepsilon$, and let $\xi(\tau)$ drop off to 0 linearly on the intervals $d < \|\tau\| < d+\varepsilon$. One easily verifies that the Fourier expansion of $\xi(\tau)$ is absolutely convergent and therefore converges to $\xi(\tau)$ uniformly. We write

(9) $$\xi(\tau) = \sum_{n=-\infty}^{\infty} c_n e(\tau n)$$

with

$$c_0 = 2d+\varepsilon .$$

Next we define

$$\xi_K(\tau) = \prod_{k=1}^{K} \xi\left(-\frac{\tau\lambda_k}{2\pi} - \theta_k\right) .$$

Clearly

(10) $$\xi_K^2(\tau) \le \xi_K(\tau)$$

and both of these functions are upper bounds for the characteristic function of $\mathcal{L}_d(T)$ (for any T). Furthermore we have

(11) $$\xi_K(\tau) = \sum_{-\infty<n_1,\ldots,n_K<\infty} c_{n_1}\cdots c_{n_K}\, e\Big(-\sum_{k=1}^{K} n_k\theta_k\Big) e\Big(-\frac{\tau}{2\pi}\sum_{k=1}^{K} n_k\lambda_k\Big)$$

$$= \sum_{\vec{n}} c(\vec{n})\, e\Big(-\frac{\tau}{2\pi}\sum_{k=1}^{K} n_k\lambda_k\Big)\ , \text{ say,}$$

where $\vec{n} = (n_1, \ldots, n_K)$ runs over all K-tuples of integers. By the linear independence of the λ_k over $\mathbb{Q}$, we see that the constant term in this expansion is

(12) $$c(\vec{0}) = c_0^K = (2d+\varepsilon)^K .$$

Since the series in (9) is absolutely convergent, so is the series in (11). Thus, given $\varepsilon_1 > 0$, there exists a number $N(\varepsilon_1)$ such that for each $N > N(\varepsilon_1)$ and all τ,

$$\Big|\xi_K(\tau) - \sum_{\|\vec{n}\|\le N} c(\vec{n})\, e\Big(-\frac{\tau}{2\pi}\sum_{k=1}^{K} n_k\lambda_k\Big)\Big| < \varepsilon_1 ,$$

where $\|\vec{n}\| = \Big(\sum_{k=1}^{K} n_k^2\Big)^{1/2}$. For such an N we have

$$\xi_K^2(\tau) \le 2\Big|\sum_{\|\vec{n}\|\le N} c(\vec{n})e(-\tfrac{\tau}{2\pi}\sum_{k=1}^{K} n_k\lambda_k)\Big|^2$$

$$+\,2\Big|\xi_K(\tau) - \sum_{\|\vec{n}\|\le N} c(\vec{n})e(-\tfrac{\tau}{2\pi}\sum_{k=1}^{K} n_k\lambda_k)\Big|^2$$

$$< \; 2\Big|\sum_{\|\vec{n}\|\le N} c(\vec{n})e(-\tfrac{\tau}{2\pi}\sum_{k=1}^{K} n_k\lambda_k)\Big|^2 + 2\varepsilon_1^2 .$$

Using this and the fact that $\xi_K^2(\tau)$ is an upper bound for the characteristic function of $\mathcal{L}_d(T)$ yields

$$(13)\quad \int_{\mathcal{L}_d(T)} |S_1(z+i\tau,\ T)|^2 d\tau \le \int_0^T \xi_K^2(\tau)|S_1(z+i\tau,\ T)|^2\, d\tau$$

$$< 2\int_0^T \Big|\sum_{\|\vec{n}\|\le N} c(\vec{n})e(-\tfrac{\tau}{2\pi}\sum_{k=1}^{K} n_k\lambda_k)\Big|^2 |S_1(z+i\tau,\ T)|^2\, d\tau$$

$$+\ 2\ \varepsilon_1^2 \int_0^T |S_1(z+i\tau,\ T)|^2\, d\tau .$$

Writing $z = x+iy$ and applying Lemma 3.4, we immediately obtain

$$(14)\quad \int_0^T |S_1(z+i\tau,\ T)|^2\, d\tau$$

$$\le (T+2\pi\delta^{-1}(\rho'))\sum_{\substack{\rho<e^{\lambda}\le\rho' \\ \lambda\varepsilon\overline{\Lambda}}} |b_\lambda(T)|^2\, e^{-2\lambda x} .$$

The first integral on the far right hand side of (13) is

$$(15)\quad \int_0^T \left| \sum_{\substack{\rho<e^{\lambda}\le\rho' \\ \lambda\in\bar{\Lambda}}} \sum_{\|\vec{n}\|\le N} b_\lambda(T)\, c(\vec{n})\, e^{-\lambda(x+iy)}\, e\Big(-\frac{\tau}{2\pi}\big(\lambda + \sum_{k=1}^{K} n_k\lambda_k\big)\Big) \right|^2 d\tau .$$

By the linear independence of Λ over $\mathbb{Q}$, we see that there is a positive constant $\delta(\rho,\rho', K, N, \Lambda)$ such that

$$\Big|\lambda + \sum_{k=1}^{K} n_k\lambda_k - \lambda' - \sum_{k=1}^{K} n_k'\lambda_k\Big| \ge \delta(\rho,\rho',K,N,\Lambda)$$

for any two frequencies $\lambda + \sum_{k=1}^{K} n_k\lambda_k$ and $\lambda' + \sum_{k=1}^{K} n_k'\lambda_k$ associated to different terms of the sum in (15). Thus, by Lemma 3.4, we see that (15) is

$$\le (T + 2\pi\delta^{-1}(\rho,\rho',K,N,\Lambda))$$

$$\cdot \sum_{\|\vec{n}\|\le N} |c(\vec{n})|^2 \sum_{\substack{\rho<e^{\lambda}\le\rho' \\ \lambda\in\Lambda}} |b_\lambda(T)|^2\, e^{-2\lambda x} .$$

From (10), (11), (12), and the linear independence of Λ over $\mathbb{Q}$ we deduce that

$$\sum_{\|\vec{n}\|\le N} |c(\vec{n})|^2 \le \sum_{\|\vec{n}\|} |c(\vec{n})|^2 = \lim_{T\to\infty} \frac{1}{T}\int_0^T \xi_K^2(\tau)\,d\tau$$

$$\le \lim_{T\to\infty} \frac{1}{T}\int_0^T \xi_K(\tau)\,d\tau$$

$$= c(\vec{0}) = (2d+\varepsilon)^K .$$

Hence (15) is

$$\leq (2d+\varepsilon)^K (T + 2\pi\delta^{-1}(\rho,\rho';K,N,\Lambda)) \sum_{\substack{\rho<e^\lambda\leq\rho' \\ \lambda\in\Lambda}} |b_\lambda(T)|^2 e^{-2\lambda x} .$$

Substituting this and (14) into (13) yields

$$\int_{\mathcal{L}_d(T)} |S_1(z+i\tau, T)|^2 d\tau < \Big(2((2d+\varepsilon)^K + \varepsilon_1^2)T + 4\pi\varepsilon_1^2\delta^{-1}(\rho')$$

$$+ 4\pi(2d+\varepsilon)^K\delta^{-1}(\rho,\rho',K,N,\Lambda)\Big) \sum_{\substack{\rho<e^\lambda\leq\rho' \\ \lambda\in\Lambda}} |b_\lambda(T)|^2 e^{-2\lambda x} .$$

Since $\rho \geq e$, $\frac{1}{2} < \sigma_1 < x < \sigma_2 < 1$, $|b_\lambda(T)| \leq 1$, and $N_\Lambda(u) << e^u$, we easily find that

$$\sum_{\substack{\rho<e^\lambda\leq\rho' \\ \lambda\in\Lambda}} |b_\lambda(T)|^2 e^{-2\lambda x} \leq \sum_{\substack{\rho<e^\lambda \\ \lambda\in\Lambda}} e^{-2\lambda\sigma_1} = \sum_{\substack{\log\rho<\lambda \\ \lambda\in\Lambda}} e^{-2\lambda\sigma_1}$$

$$= \int_{\log\rho}^{\infty} e^{-2\sigma_1 u} dN_\Lambda(u) = N_\Lambda(u)e^{-2\sigma_1 u}\Big|_{\log\rho}^{\infty}$$

$$+ 2\sigma_1 \int_{\log\rho}^{\infty} N_\Lambda(u)e^{-2\sigma_1 u} du << \rho^{1-2\sigma_1} ,$$

where the $<<$ depends on σ_1 and Λ . Finally, we have

$$(16) \quad \int_{\mathcal{L}_d(T)} |S_1(z+i\tau, T)|^2 d\tau << \rho^{1-2\sigma_1} \Big(((2d+\varepsilon)^K + \varepsilon_1^2)T$$

$$+ \varepsilon_1^2\delta^{-1}(\rho') + (2d+\varepsilon)^K\delta^{-1}(\rho,\rho',K,N,\Lambda)\Big) .$$

The estimation of the second integral on the right-hand side of (8) is much simpler. Again writing $z = x+iy$ and using Lemma 3.4, we have

$$\int_{\mathcal{L}_d(T)} |S_2(z+i\tau,\ T)|^2 d\tau \le \int_0^T |S_2(z+i\tau,\ T)|^2 d\tau$$

$$\le (T + 2\pi\delta^{-1}(T)) \sum_{\substack{\rho' < e^\lambda \le T \\ \lambda \varepsilon \Lambda}} |b_\lambda(T)|^2 e^{-2\lambda x} .$$

Estimating as above, we see that

$$\sum_{\substack{\rho' < e^\lambda \le T \\ \lambda \varepsilon \Lambda}} |b_\lambda(T)|^2 e^{-2\lambda x} << (\rho')^{1-2\sigma_1} ,$$

where the implicit constant depends on σ_1 and Λ. Thus

$$(17) \int_{\mathcal{L}_d(T)} |S_2(z+i\tau,\ T)|^2 d\tau << (\rho')^{1-2\sigma_1}(T + \delta^{-1}(T)) .$$

Combining the results of (7), (8), (16), and (17) gives

$$(18) \int_{\mathcal{L}_d(T)} \max_{s\varepsilon C} |(S(s+i\tau,T)|^2 d\tau << \rho^{1-2\sigma_1}\Big(((2d+\varepsilon)^K + \varepsilon_1^2)T$$

$$+ \varepsilon_1^2\delta^{-1}(\rho') + (2d+\varepsilon)^K\delta^{-1}(\rho,\rho',K,N,\Lambda)\Big)$$

$$+ (\rho')^{1-2\sigma_1}(T + \delta^{-1}(T)) ,$$

where the $<<$ depends on δ, A, σ_1, and Λ, or, since δ and A depend only on σ_1, σ_2, and C, the $<<$ depends on σ_1 , σ_2 , C , and Λ . We now choose ρ' so large that

$$(\rho')^{1-2\sigma_1} \leq (2d)^K \rho^{1-2\sigma_1} ,$$

and ε and ε_1 so small that

$$(2d+\varepsilon)^K + \varepsilon_1^2 \leq 2(2d)^K .$$

It follows that for all sufficiently large T , the right-hand side of (18) is

$$<< (2d)^K \rho^{1-2\sigma_1}(T + \delta^{-1}(T)) .$$

This proves the lemma. □

Lemma 3.7. Assume the same notation and hypotheses as in Lemma 3.6 except that instead of one sequence of functions $\{b_\lambda(T)\}_{\lambda\varepsilon\Lambda}$ we have N such sequences $\{b_\lambda^{(n)}(T)\}_{\lambda\varepsilon\Lambda}$, $1 \leq n \leq N$, with $|b_\lambda^{(n)}(T)| \leq 1$. Furthermore, assume that $\delta(T) >> \frac{1}{T}$. If for each $T \geq 0$, $\mathcal{J}(T)$ is a subset of [0,T] with measure

$$J(T) = T(1+o(1)) ,$$

then for all large T , there is a $\tau \varepsilon \mathcal{J}(T)$ such that

$$\left\|- \frac{\tau^{\lambda_k}}{2\pi} - \theta_k\right\| \leq d , \qquad 1 \leq k \leq K$$

and

$$\max_{1\leq n\leq N}\ \max_{s\varepsilon C}\left|\sum_{\substack{\rho<e^{\lambda}\leq T\\ \lambda\varepsilon\Lambda}} b_{\lambda}^{(n)}(T)e^{-\lambda(s+i\tau)}\right| << \rho^{1/2-\sigma_1} .$$

The constant implicit in $<<$ depends on σ_1 , σ_2 , C , Λ , and N .

<u>Proof</u>: Let

$$S_n(s+i\tau,\ T) = \sum_{\substack{\rho<e^{\lambda}\leq T\\ \lambda\varepsilon\Lambda}} b_{\lambda}^{(n)}(T)e^{-\lambda(s+i\tau)} \ ,\ 1 \leq n \leq N$$

$$MS(\tau,T) = \max_{1\leq n\leq N}\ \max_{s\varepsilon C}|S_n(s+i\tau,\ T)| \ .$$

We write

$$\mathcal{H}_d(T) = \{\tau\ \varepsilon\ \mathcal{D}_d(T)\,|\,MS(\tau,T) \leq c\rho^{1/2-\sigma_1}\}$$

and denote the measure of $\mathcal{H}_d(T)$ by $G_d(T)$. We will be done if we show that for some choice of c (depending at most on σ_1 , σ_2 , C , Λ , and N) and for all large T , the set

$$\mathcal{H}_d(T) \cap \mathcal{J}(T)$$

is non-empty. Now clearly

$$(MS(\tau,T))^2 \le \left(\sum_{n=1}^{N} \max_{s\varepsilon C} |S_n(s+i\tau,\ T)| \right)^2$$

$$\le \ N \sum_{n=1}^{N} \max_{s\varepsilon C} |S_n(s+i\tau,\ T)|^2 \ .$$

Hence

$$\int_{\mathcal{I}_d(T)} (MS(\tau,T))^2 d\tau \le N \sum_{n=1}^{N} \left(\int_{\mathcal{I}_d(T)} \max_{s\varepsilon C} |S_n(s+i\tau,\ T)|^2 \, d\tau \right) .$$

By Lemma 3.6 and the hypothesis $\delta(T) \gg \frac{1}{T}$ (here the implicit constant depends only on Λ), we find for T large enough

$$\int_{\mathcal{I}_d(T)} (MS(\tau,T))^2 d\tau \ll \rho^{1-2\sigma_1} (2d)^K T \ ,$$

where the implied constant depends on σ_1 , σ_2 , C , Λ , and N . From this it follows that the measure of the complement of $\mathcal{G}_d(T)$ in $\mathcal{I}_d(T)$ is

$$\ll \frac{(2d)^K}{c^2} T \ .$$

Thus, choosing c large enough as a function of σ_1 , σ_2 , C , Λ , and N , we can ensure that

$$G_d(T) > I_d(T) - \tfrac{1}{4}(2d)^K T \ ,$$

where $I_d(T)$ is the measure of $\mathcal{L}_d(T)$. By Lemma 3.5, if T is large enough,

$$I_d(T) > \tfrac{1}{2}(2d)^K T .$$

Therefore

$$G_d(T) > \tfrac{1}{4}(2d^K)T .$$

Since

$$J(T) = T(1 + o(1)) ,$$

$\mathcal{J}(T)$ and $\mathcal{H}_d(T)$ must overlap for large T. This gives the result.

§3. Three Lemmas on L-functions

The following is a simple adaptation of Titchmarsh [19; Theorem 14.20].

Lemma 3.8. For $x > 1$,

$$(19)\quad \frac{L'}{L}(s,\chi) = -\sum_{n<x^2} \Lambda_x(n)\chi(n)n^{-s} + E(\chi)\,\frac{x^{2(1-s)} - x^{(1-s)}}{(1-s)^2 \log x}$$

$$+ \frac{1}{\log x}\sum_{r=0}^{\infty} \frac{x^{-2r-\alpha-s} - x^{-2(2r-\alpha-s)}}{(2r+\alpha+s)^2}$$

$$+ \frac{1}{\log x} \sum_{\rho} \frac{x^{\rho-s} - x^{2(\rho-s)}}{(s-\rho)^2}$$

where

$$E(\chi) = \begin{cases} 1 & \text{if } \chi \text{ is the principal character} \\ 0 & \text{otherwise} \end{cases} ,$$

$$\sigma = \begin{cases} 1 & \text{if } \chi(-1) = -1 \\ 0 & \text{if } \chi(-1) = 1 \end{cases} ,$$

and

$$\Lambda_x(n) = \begin{cases} \Lambda(n) & \text{if } n \le x \\ \frac{\Lambda(n)\log(x^2/n)}{\log n} & \text{if } x < n \le x^2 \\ 0 & \text{if } x^2 < n \end{cases} .$$

<u>Proof</u>: Let $\alpha = \max(2, 1+\sigma)$. Then

$$\frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} \frac{x^{z-s} - x^{2(z-s)}}{(z-s)^2} \frac{L'}{L}(z,\chi)\,dz$$

$$= -\frac{1}{2\pi i} \sum_{n=1}^{\infty} \Lambda(n)\chi(n) \int_{\alpha-i\infty}^{\alpha+i\infty} \frac{x^{z-s} - x^{2(z-s)}}{(z-s)^2\, n^z}\,dz$$